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INDECOMPOSABLE GENERALIZED WEIGHT MODULES OVER THE ALGEBRA OF POLYNOMIAL INTEGRO-DIFFERENTIAL OPERATORS

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ABSTRACT. For the algebra $\mathbb{I}_1 = K\langle x, \frac{d}{dx}, \int\rangle$ of polynomial integro-differential operators over a field K of characteristic zero, a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length is given. Each such module is an infinite dimensional uniserial module. Ext-groups are found between indecomposable generalized weight modules, it is proven that they are finite dimensional vector spaces.

Key Words: the algebra of polynomial integro-differential operators, generalized weight module, indecomposable module, simple module.

Mathematics subject classification 2000: 16D60, 16D70, 16P50, 16U20.

1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; $\mathbb{N}_+ := \{1, 2, \dots\}$ and $\mathbb{Z}_{\leq 0} := -\mathbb{N}$; K is a field of characteristic zero and K^* is its group of units; $P_1 := K[x]$ is a polynomial algebra in one variable x over K; $\partial := \frac{d}{dx}$; $\text{End}_K(P_1)$ is the algebra of all K-linear maps from P_1 to P_1, and $\text{Aut}_K(P_1)$ is its group of units (i.e. the group of all the invertible linear maps from P_1 to P_1); the subalgebras $A_1 := K\langle x, \partial\rangle$ and $\mathbb{I}_1 := K\langle x, \partial, \int\rangle$ of $\text{End}_K(P_1)$ are called the (first) *Weyl algebra* and the *algebra of polynomial integro-differential operators* respectively where $\int : P_1 \to P_1$, $p \mapsto \int p\, dx$, is the *integration*, i.e. $\int : x^n \mapsto \frac{x^{n+1}}{n+1}$ for all $n \in \mathbb{N}$. The algebra $\mathbb{I}_1$ is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals, [2].

In Section 2, a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length is given (Theorem 2.5). A similar classification is given in [1] for the generalized Weyl algebras where a completely different approach was taken. Properties of the algebras $\mathbb{I}_n := \mathbb{I}_1^{\otimes n}$ of polynomial integro-differential operators in arbitrary many variables are studied in [2] and [5]. The groups $\text{Aut}_{K-\text{alg}}(\mathbb{I}_n)$ are found in [3]. The simple $\mathbb{I}_1$-modules are classified in [4].

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2. Classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length

In this section, a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length is given (Theorem 2.5).

As an abstract algebra, the algebra $\mathbb{I}_1$ is generated by the elements ∂, $H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations, [2, Proposition 2.2] (where $[a, b] := ab - ba$):

$$\partial \int = 1, \ \ [H, \int] = \int, \ \ [H, \partial] = -\partial, \ \ H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra $\mathbb{I}_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \ \ i, j \in \mathbb{N}, \tag{1}$$

satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00}\partial^j$. The matrices of the linear maps $e_{ij} \in \mathrm{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s\in\mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \mathrm{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js}x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!}E_{ij}, \tag{2}$$

$Ke_{ij} = KE_{ij}$, and $F := \bigoplus_{i,j\geq 0} Ke_{ij} = \bigoplus_{i,j\geq 0} KE_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices.

$\mathbb{Z}$-grading on the algebra $\mathbb{I}_1$ and the canonical form of an integro-differential operator, [2]. The algebra $\mathbb{I}_1 = \bigoplus_{i\in\mathbb{Z}} \mathbb{I}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{I}_{1,i}\mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|}D_1 = D_1\partial^{|i|} & \text{if } i < 0, \end{cases} \tag{3}$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of $\mathbb{I}_1$, $He_{ii} = e_{ii}H = (i + 1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals of D_1); $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; ${}_{D_1}(D_1\partial^i) \simeq D_1$, $d\partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \to D_1 \int^i$, $d \mapsto d \int^i$, and $\partial^i \cdot : D_1 \to \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$.

Each element a of the algebra $\mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i>0} a_{-i}\partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij}e_{ij} \tag{4}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [2].

Let $v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$

Then $\mathbb{I}_{1,i} = D_1v_i = v_iD_1$ and an element $a \in \mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i\in\mathbb{Z}} b_iv_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij}e_{ij} \tag{5}$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set $\{H^j\partial^i, H^j, \int^i H^j, e_{st} \,|\, i \geq 1; j, s, t \geq 0\}$ is a K-basis for the algebra $\mathbb{I}_1$. The multiplication in the algebra $\mathbb{I}_1$ is given by the rule:

$$\int H = (H - 1)\int, \quad H\partial = \partial(H - 1), \quad \int e_{ij} = e_{i+1,j},$$

$$e_{ij}\int = e_{i,j-1}, \quad \partial e_{ij} = e_{i-1,j}, \quad e_{ij}\partial = \partial e_{i,j+1},$$

$$He_{ii} = e_{ii}H = (i + 1)e_{ii}, \quad i \in \mathbb{N},$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$.

The algebra $\mathbb{I}_1$ has the only proper ideal $F = \bigoplus_{i,j\in\mathbb{N}} Ke_{ij} \simeq M_\infty(K)$ and $F^2 = F$. The factor algebra $\mathbb{I}_1/F$ is canonically isomorphic to the skew Laurent polynomial algebra $B_1 := K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H + 1$, via $\partial \mapsto \partial$, $\int \mapsto \partial^{-1}$, $H \mapsto H$ (where $\partial^{\pm 1}\alpha = \tau^{\pm 1}(\alpha)\partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1}H$).

An $\mathbb{I}_1$-module M is called a *weight* module if $M = \oplus_{\lambda\in K} M_\lambda$ where $M_\lambda := \{m \in M \mid Hm = \lambda m\}$. An $\mathbb{I}_1$-module M is called a *generalized weight* module if $M = \oplus_{\lambda\in K} M^\lambda$ where $M^\lambda := \{m \in M \mid (H-\lambda)^n m = 0 \text{ for some } n = n(m)\}$. The set $\mathrm{Supp}(M) := \{\lambda \in K \mid M^\lambda \neq 0\}$ is called the *support* of the generalized weight module M. For all $\lambda \in K$ and $n \geq 1$,

$$\partial^n M^\lambda \subseteq M^{\lambda-n} \text{ and } \int^n M^\lambda \subseteq M^{\lambda+n}.$$

Let $0 \to N \to M \to L \to 0$ be a short exact sequence of $\mathbb{I}_1$-modules. Then M is a generalized weight module iff so are the modules N and L, and in this case

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(L).$$

For each $\mathbb{I}_1$-module M, there is a short exact sequence of $\mathbb{I}_1$-modules

$$0 \to FM \to M \to \overline{M} := M/FM \to 0 \tag{6}$$

where
(i) $F \cdot FM = FM$, and
(ii) $F \cdot \overline{M} = 0$,
and the properties (i) and (ii) determine the short exact sequence (6) uniquely, i.e. if $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of $\mathbb{I}_1$-modules such that $FM_1 = M_1$ and $FM_2 = 0$ then $M_1 \cong FM$ and $M_2 \simeq \overline{M}$.

Notice that

$$FM \simeq K[x]^I, \tag{7}$$

i.e. the $\mathbb{I}_1$-module FM is isomorphic to the direct sum of I copies of the simple weight $\mathbb{I}_1$-module $K[x]$. Clearly, $\overline{M}$ is a B_1-module.

The indecomposable $\mathbb{I}_1$-modules $M(n,\lambda)$. For $\lambda \in K$ and a natural number $n \geq 1$, consider the B_1-module

$$M(n,\lambda) := B_1 \otimes_{K[H]} K[H]/(H-\lambda)^n. \tag{8}$$

Clearly,

$$M(n,\lambda) \simeq B_1/B_1(H-\lambda)^n \simeq \mathbb{I}_1/(F + \mathbb{I}_1(H-\lambda)^n). \tag{9}$$

The $\mathbb{I}_1$-module/B_1-module $M(n,\lambda)$ is a generalized weight module with $\mathrm{Supp} M(n,\lambda) = \lambda + \mathbb{Z}$,

$$M(n,\lambda) = \bigoplus_{i\in\mathbb{Z}} M(n,\lambda)^{\lambda+i} \text{ and } \dim M(n,\lambda)^{\lambda+i} = n \text{ for all } i \in \mathbb{Z}. \tag{10}$$

For an algebra A, we denote by $A - \mathrm{Mod}$ its module category. The next proposition describes the set of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length M with $FM = 0$.

Proposition 2.1. (1) *$M(n,\lambda)$ is an indecomposable, generalized weight $\mathbb{I}_1$-module of finite length n.*
(2) *$M(n,\lambda) \cong M(m,\mu)$ if and only if $n = m$ and $\lambda - \mu \in \mathbb{Z}$.*
(3) *Let M be a generalized weight B_1-module of length n (i.e. let M be a generalized weight $\mathbb{I}_1$-module such that $FM = 0$, by (6)). Then M is indecomposable if and only if $M \simeq M(n,\lambda)$ for some $\lambda \in K$.*

Proof. 1. Since $(B_1)_{K[H]} = \oplus_{i\in\mathbb{Z}} \partial^i K[H]$ is a free right $K[H]$-module, the functor

$$B_1 \otimes_{K[H]} - : K[H] - \mathrm{Mod} \to B_1 - \mathrm{Mod}, \quad N \mapsto B_1 \otimes_{K[H]} N,$$

is an exact functor. The $K[H]$-module $K[H]/(H-\lambda)^n$ is an indecomposable, hence the B_1-module $M(n,\lambda)$ is indecomposable and generalized weight of length n.

2. ($\Rightarrow$) Suppose that $\mathbb{I}_1$-modules $M(n,\lambda)$ and $M(m,\mu)$ are isomorphic. Then $\mathrm{Supp}(M(n,\lambda)) = \mathrm{Supp}(M(n,\lambda))$, i.e. $\lambda + \mathbb{Z} = \mu + \mathbb{Z}$, i.e. $\lambda - \mu \in \mathbb{Z}$. Then $n = m$, by (10).

($\Leftarrow$) Suppose that $k := \lambda - \mu \in \mathbb{Z}$ and $n = m$. We may assume that $k \geq 1$. Using the equality $(H-\lambda)^n \partial^k = \partial^k (H - \lambda - k)^n = \partial^k (H-\mu)^n$, we see that the B_1-homomorphism

$$M(n,\lambda) = B_1/B_1(H-\lambda)^n \to M(n,\mu) = B_1/B_1(H-\mu)^n, \;\; 1 + B_1(H-\lambda)^n \mapsto \partial^k + B_1(H-\mu)^n,$$

is an isomorphism with the inverse given by the rule $1 + B_1(H-\mu)^n \mapsto \partial^{-k} + B_1(H-\lambda)^n$.

3. ($\Leftarrow$) This implication follows from statement 2.

($\Rightarrow$) Each indecomposable, generalized weight B_1-module M is of the type $B_1 \otimes_{K[H]} N$ for an indecomposable $K[H]$-module N of length n. Notice that $N \simeq K[H]/(H-\lambda)^n$ for some $\lambda \in K$. Therefore, $M \simeq M(n, \lambda)$. $\Box$

Lemma 2.2. *Let M be an indecomposable, generalized weight $\mathbb{I}_1$-module. Then* $\mathrm{Supp}(M) \subseteq \lambda + \mathbb{Z}$ *for some $\lambda \in K$.*

Proof. Let $M = \oplus_{\mu \in \mathrm{Supp}(M)} M^\mu$ be a generalized weight $\mathbb{I}_1$-module. Then

$$M = \bigoplus_{\mu+\mathbb{Z} \in \mathrm{Supp}(M)/\mathbb{Z}} M_{\mu+\mathbb{Z}}$$

is a direct sum of $\mathbb{I}_1$-submodules $M_{\mu+\mathbb{Z}} := \oplus_{i \in \mathbb{Z}} M^{\mu+i}$ where $\mathrm{Supp}(M)/\mathbb{Z}$ is the image of the support $\mathrm{Supp}(M)$ under the abelian group epimorphism $K \to K/\mathbb{Z}$, $\gamma \mapsto \gamma + \mathbb{Z}$. The $\mathbb{I}_1$-module M is indecomposable, hence $M = M_{\lambda+\mathbb{Z}}$ for some $\lambda \in K$, i.e. $\mathrm{Supp}(M) \subseteq \lambda + \mathbb{Z}$. $\Box$

The next lemma describes the set of indecomposable, generalized weight $\mathbb{I}_1$-modules M with $FM = M$.

Lemma 2.3. *Let M be an indecomposable, generalized weight $\mathbb{I}_1$-modules M. Then the following statements are equivalent.*

1. *$FM = M$.*
2. *$M \simeq K[x]$.*
3. *$\mathrm{Supp}(M) \subseteq \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) : If $FM = M$ then $M \simeq K[x]^{(I)}$ for some set I necessarily with $|I| = 1$ since M is indecomposable, i.e. $M \simeq K[x]$.

(2) $\Rightarrow$ (3) : $\mathrm{Supp}(K[x]) = \{1, 2, \ldots\} \subseteq \mathbb{N}$.

(3) $\Rightarrow$ (1) : Suppose that $\mathrm{Supp}(M) \subseteq \mathbb{N}$. Using the short exact sequence of $\mathbb{I}_1$-modules $0 \to FM \to M \to \overline{M} := M/FM \to 0$ we see that $\mathrm{Supp}(M) = \mathrm{Supp}(FM) \cup \mathrm{Supp}(\overline{M})$. Since $\mathrm{Supp}(FM) = \mathrm{Supp}(K[x]^{(I)}) = \{1, 2, \ldots\}$ and $\mathrm{Supp}(\overline{M})$ is an abelian group, we must have $\overline{M} = 0$ (since $\mathrm{Supp}(M) \subseteq \mathbb{N}$), i.e. $M = FM$. $\Box$

The following result is a key step in obtaining a classification of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length.

Theorem 2.4. *Let M be a generalized weight $\mathbb{I}_1$-module of finite length. Then the short exact sequence (6) splits.*

Proof. We can assume that $FM \neq 0$ and $\overline{M} \neq 0$. It is obvious that $FM \simeq K[x]^s$ for some $s \geq 1$ and the B_1-module $\overline{M} \simeq \bigoplus_{i=1}^t M(n_i, \lambda_i)$ for some $n_i \geq 1$, $\lambda_i \in K$ and $t \geq 1$. It suffices to show that

$$\mathrm{Ext}^1_{\mathbb{I}_1}(M(n, \lambda), K[x]) = 0 \tag{11}$$

for all $n \geq 1$ and $\lambda \in K$. If $\lambda \in \mathbb{Z}$ we can assume that $\lambda = 0$, by Proposition 2.1.(2).

(i) $F(H-\lambda)^n = F$: The equality follows from the equalities $e_{ij}(H-\lambda)^n = e_{ij}(j+1-\lambda)$ and the choice of λ.

(ii) $M(n, \lambda) = \mathbb{I}_1/\mathbb{I}_1(H-\lambda)^n$: By (i), $\mathbb{I}_1(H-\lambda)^n \supseteq F(H-\lambda)^n = F$. Hence,

$$M(n, \lambda) = \mathbb{I}_1/(F + \mathbb{I}_1(H-\lambda)^n) = \mathbb{I}_1/\mathbb{I}_1(H-\lambda)^n.$$

(iii) *The equality (11) holds*: Let $M = M(n, \lambda)$. By (ii), the short exact sequence of $\mathbb{I}_1$-modules

$$0 \to \mathbb{I}_1(H-\lambda)^n \to \mathbb{I}_1 \to M \to 0 \tag{12}$$

is a projective resolution of the $\mathbb{I}_1$-module M since the map

$$\cdot(H-\lambda)^n : \mathbb{I}_1 \to \mathbb{I}_1(H-\lambda)^n, \ \ a \mapsto a(H-\lambda)^n,$$

is an isomorphism of $\mathbb{I}_1$-modules, by the choice of λ. Then

$$\mathrm{Ext}^1_{\mathbb{I}_1}(M, K[x]) \simeq Z^1/B^1$$

where $Z^1 = \mathrm{Hom}_{\mathbb{I}_1}(\mathbb{I}_1(H-\lambda)^n, K[x]) \simeq K[x]$ and $B^1 \simeq (H-\lambda)^n K[x] = K[x]$, by the choice of λ. Hence, the equality (11) holds. The proof of the theorem is complete. $\Box$

The next theorem is a classification of the set of indecomposable, generalized weight $\mathbb{I}_1$-modules of finite length.

Theorem 2.5. *Each indecomposable, generalized weight $\mathbb{I}_1$-module of finite length is isomorphic to one of the modules below:*

(1) $K[x]$,
(2) $M(n,\lambda)$ *where* $n \geq 1$ *and* $\lambda \in \Lambda$ *where* Λ *is any fixed subset of* K *such that the map* $\Lambda \to (K/\mathbb{Z})$, $\lambda \mapsto \lambda + \mathbb{Z}$, *is a bijection.*

The $\mathbb{I}_1$-modules above are pairwise non-isomorphic, indecomposable, generalized weight and of finite length.

Proof. The theorem follows from Theorem 2.4, Proposition 2.1 and Lemma 2.3. □

Corollary 2.6. *Every indecomposable, generalized weight $\mathbb{I}_1$-module is an uniserial module.*

Proof. The statement follows from Theorem 2.5.□

Homomorphisms and Ext-groups between indecomposables.

Proposition 2.7. (1) *Let M and N be generalized weight $\mathbb{I}_1$-modules such that* $\mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \emptyset$. *Then* $\mathrm{Hom}_{\mathbb{I}_1}(M,N) = 0$.
(2) $\mathrm{Hom}_{\mathbb{I}_1}(M(n,\lambda), K[x]) = 0$.
(3) $\mathrm{Hom}_{\mathbb{I}_1}(K[x], M(n,\lambda)) = 0$.
(4) $\mathrm{Hom}_{\mathbb{I}_1}(M(n,\lambda), M(m,\lambda)) \simeq \mathrm{Hom}_{K[H]}(K[H]/((H-\lambda)^n), (K[H]/((H-\lambda)^m)) \simeq K[H]/((H-\lambda)^{\min(n,m)})$.

Proof. 1. Statement 1 is obvious.

2. Statement 2 follows from the fact that $FM(n,\lambda) = 0$ and $Fp = K[x]$ for all nonzero elements $p \in K[x]$ (since $K[x]$ is a simple $\mathbb{I}_1$-module, F is an ideal of the algebra $\mathbb{I}_1$ such that $FK[x] = K[x]$).

3. Statement 3 follows from the fact that $FK[x] = K[x]$ and $FM(n,\lambda) = 0$: $f(K[x]) = f(FK[x]) = Ff(K[x]) = 0$ for any $f \in \mathrm{Hom}_{\mathbb{I}_1}(K[x], M(n,\lambda))$.

4. The first isomorphism is obvious. Then the second isomorphism follows. □

Proposition 2.8. (1) $\mathrm{Ext}^1_{\mathbb{I}_1}(K[x], K[x]) = 0$.
(2) $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), K[x]) = 0$.
(3) $\mathrm{Ext}^1_{\mathbb{I}_1}(K[x], M(n,\lambda)) = 0$.
(4) $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), M(m,\mu)) = \begin{cases} K & \text{if } \lambda - \mu \in \mathbb{Z}, \\ 0 & \text{if } \lambda - \mu \notin \mathbb{Z}. \end{cases}$

Proof. 1. Let $0 \to K[x] \to N \to K[x] \to 0$ be a s.e.s. of $\mathbb{I}_1$-modules. Then $FN = N$ (since $FK[x] = K[x]$), and so N is an epimorphic image of the semisimple $\mathbb{I}_1$-module $F \oplus F$. Hence, $N \simeq K[x] \oplus K[x]$ (since ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$).

2. See (11).

3. Let $0 \to M = M(n,\lambda) \to L \to K[x] \to 0$ be a s.e.s. of $\mathbb{I}_1$-modules. Since $FM = 0$, we have $FL = FK[x] \simeq K[x]$ is a submodule of L such that $FL \cap M = 0$ (since otherwise $FL \subseteq M$ by simplicity of the $\mathbb{I}_1$-module $FL \simeq K[x]$, and so $0 \neq K[x] \simeq FL = F^2L \subseteq FM = 0$, a contradiction). Then $FL \oplus M \subseteq L$. Furthermore, $FL \oplus M = L$ since $l_{\mathbb{I}_1}(FL \oplus M) = l_{\mathbb{I}_1}(L)$. This means that the s.e.s. splits.

4. Let $0 \to M_1 \to M \to M_1 \to 0$ be a s.e.s. of generalized weight $\mathbb{I}_1$-modules. If $\mathrm{Supp}(M_1) \cap \mathrm{Supp}(M_2) = \emptyset$, it splits. In particular, $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), M(m,\mu)) = 0$ if $\lambda - \mu \notin \mathbb{Z}$. If $\lambda - \mu \in \mathbb{Z}$ we can assume that $\lambda = \mu$ (sine $M(m,\lambda) \simeq M(m,\mu)$). Using (12), where we assume that $\lambda = 0$ if $\lambda \in \mathbb{Z}$, we see that $\mathrm{Ext}^1_{\mathbb{I}_1}(M(n,\lambda), M(m,\lambda)) \simeq M(m,\lambda)/(H-\lambda)M(m,\lambda) \simeq K$. □

Since the left global dimension of the algebra $\mathbb{I}_1$ is 1, [6], Proposition 2.7 and Proposition 2.8 describe all the Ext-groups between indecomposable, generalized weight $\mathbb{I}_1$-modules. This is also obvious from the proofs of the propositions.

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